EXPANSION OF A TRI-AXIAL GAS ELLIPSOID IN A REGULAR BEHAVIOR

(RAZLET TREKHOSNOGO GAZOVOGO ELLIPSOIDA V REGULIARNOM REZHIME)

PMM Vol.29, № 1, 1965, pp. 134-140

I.V. NEMCHINOV
(Moscow)

(Received July 2, 1964)

Exact particular solutions of the equations of gas dynamics as given by Ovsiannikov [1] are used to describe adiabatic expansion of gas ellipsoid in vacuum. Numerical results describing the variation of shape of a gas cloud and the time history of its expansion are given.

Similar solutions for the case of gas motion with heating were found. These solutions are a generalization of the particular solutions of the Sedov's [2 and 3] problem of one-dimensional adiabatic expansion and similar solutions [4 and 5] for motion with heating (where the velocity is proportional to the distance from the center of symmetry). The pressure (and density) distribution turns out to depend on one arbitrary function, where the surfaces corresponding to constant values of the function are ellipsoids.

1. We shall write the equation of gas dynamics referred to Lagrangian coordinates (the initial position of a particle)

$$\frac{\partial x_i}{\partial t} = u_i, \quad \frac{\partial u_i}{\partial t} = -v\frac{\partial p}{\partial x_i} \quad (i = 1, 2, 3), \quad \frac{v(\xi_k, t)}{v(\xi_k, 0)} = \frac{\partial x_1 \partial x_2 \partial x_3}{\partial \xi_1 \partial \xi_2 \partial \xi_3} \quad \left(v = \frac{1}{\rho}\right) \tag{1.1}$$

where u_i is the velocity of particles, t is the time, p is the pressure, v is the specific volume, ρ is the density and x_i are Eulerian coordinates of the particles whose Lagrangian coordinates $\xi_i = x_i(0)$; the partial derivatives with respect to time in the equation describing velocity of particles and in the momentum equation are taken at constant values of ξ_1, ξ_2, ξ_3, and the Jacobian of the transformation from x_i to ξ_i in the right-hand side of the continuity equation is taken at t = const . The notation $f(\xi_k)$ represents $f(\xi_1, \xi_2, \xi_3)$.

It is evident that the relationship

$$\frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_i} + \frac{\partial p}{\partial \xi_2}\frac{\partial \xi_2}{\partial x_i} + \frac{\partial p}{\partial \xi_3}\frac{\partial \xi_3}{\partial x_i} \tag{1.2}$$

also holds.

In order to describe the motion in regular conditions it is necessary to separate the variables (the Lagrangian space coordinates and the time). Then, the third of the equations (1.1) and Equation (1.2) are simplified (1.3)

$$x_i = x_i^\circ(t)\, X(\xi_k), \quad u_i = u_i^\circ(t)\, U(\xi_k), \quad p = p^\circ(t)\, P(\xi_k), \quad v = v^\circ(t) V(\xi_k)$$

Without loss of generality, we can set $x_i^\circ(0) = 1$, from which follows $X(\xi_k) = x_i^\circ(0) = \xi_i$. Introducing the notation $x_i^\circ(t) = \varphi_i(t)$, we obtain

$$x_i = \varphi_i \xi_i, \quad u_i = \xi_i \frac{d\varphi_i}{dt} = \frac{x_i}{\varphi_i}\frac{d\varphi_i}{dt}, \quad \frac{\partial x_1\, \partial x_2\, \partial x_3}{\partial \xi_1 \partial \xi_2 \partial \xi_3} = \varphi_1\varphi_2\varphi_3, \quad \frac{\partial p}{\partial x_i} = \frac{\partial P}{\partial \xi_i}\frac{p^\circ(t)}{\varphi_i(t)} \tag{1.4}$$

By substituting (1.3) into (1.1), a system of ordinary differential equations in φ_0 and a system of partial differential equations of first order in P is obtained

$$\varphi_i \frac{d^2\varphi_i}{dt^2}\frac{1}{p^\circ(t)\, v^\circ(t)} = -\frac{V}{\xi_i}\frac{\partial P}{\partial \xi_i} = \alpha_i \tag{1.5}$$

In order to integrate (1.5) it is necessary to prescribe a relation between $p^\circ(t)$ and $v^\circ(t)$ as well as the dependence of P on V and ξ which can be found from the energy equation provided it admits the separation of variables (1.3). Moreover, it is necessary to prescribe the value of the separation constants α_i, the initial values $p^\circ(0)$ and $v^\circ(0)$, as well as the value of P at the arbitrary point. Without loss of generality we can set $P(0, 0, 0) = V(0, 0, 0) = 1$, then $p^\circ(t)$ and $v^\circ(t)$ are the pressure and the specific volume at the center of coordinates.

2. We shall consider the adiabatic flow of a gas

$$p(\xi_k, t)\, v^\varkappa(\xi_k, t) = p^\circ(t)\, [v^\circ(t)]^\varkappa P V^\varkappa \tag{2.1}$$

The initial distribution of entropy can be prescribed by an arbitrary function

$$PV^\varkappa = f^{-\varkappa}(\xi_k), \qquad f(0, 0, 0) = 1 \tag{2.2}$$

Then the system of equations (1.5) assumes the form

$$\varphi_i \frac{d^2\varphi_i}{dt^2}(\varphi_1\varphi_2\varphi_3)^{-(\varkappa-1)} = \alpha_i p^\circ(0)\, v^\circ(0), \qquad v^\circ(t) = v^\circ(0)\varphi_1\varphi_2\varphi_3 \tag{2.3}$$

$$\frac{\partial P}{\partial \xi_i}\frac{P^{-1/\varkappa}}{\xi_i}\frac{1}{f(\xi_k)} = -\alpha_i \tag{2.4}$$

The values of the separation constants α_i can be determined from the given dimensions of the gas cloud along the axes ξ_i^*. In fact, the partial differential equations (2.4) can be integrated along the respective axes in the same manner as ordinary equations, provided the two remaining coordinates are kept constant. For example, integrating along the axis ξ_1, we get

$$P^{\frac{\varkappa-1}{\varkappa}}(\xi_1, \xi_2 = 0, \xi_3 = 0) = 1 - \frac{\varkappa-1}{\varkappa}\alpha_1 \int_0^{\xi_1} f(\xi_1, 0, 0)\, \xi_1 d\xi_1 \tag{2.5}$$

Satisfying the boundary condition $P = 0$ on the interface with the vacuum, the α_i for corresponding ξ_i^* are found. For $f = 1$ the simple equation results

$$P^{\frac{\varkappa-1}{\varkappa}} = 1 - (\alpha_1 \xi_1^2 + \alpha_2\xi_2^2 + \alpha_3\xi_3^2)\left(\frac{\varkappa-1}{2\varkappa}\right) \tag{2.6}$$

Hence, provided all $\alpha_i > 0$, then the boundary surface is a tri-axial ellipsoid with semiaxes given by

$$\alpha_i = \frac{2\varkappa}{(\varkappa - 1)} \frac{1}{(\xi_i^*)^2} \tag{2.7}$$

The successive integration along the axes can be used in the general case. In order to have the results independent of the order of integration it is necessary to impose the following condition of the function f:

$$\frac{1}{\alpha_1 \xi_1} \frac{\partial f}{\partial \xi_1} = \frac{1}{\alpha_2 \xi_2} \frac{\partial f}{\partial \xi_2} = \frac{1}{\alpha_3 \xi_3} \frac{\partial f}{\partial \xi_3} \tag{2.8}$$

When the condition (2.8) is satisfied, the system (2.4) is compatible.

It follows from (2.8) that f is invariant with respect to exchange of $\alpha_i \xi_i^2$ for $\alpha_i \xi_i^2$ which takes place when f is a function of a linear combination of squares of the Lagrangian coordinates

$$f = f(\sigma), \qquad \sigma = \eta^2 = \alpha_1 \xi_1^2 + \alpha_2 \xi_2^2 + \alpha_3 \xi_3^2 \tag{2.9}$$

By changing the arbitrary function f, different pressure distribution profiles within the ellipsoid can be obtained. Let $P = \mathscr{P}(\sigma)$, then $\Omega = -2\mathscr{P}'(\sigma)$, where Ω is the nondimensional density profile, $V\Omega = 1$. We shall consider, for example, pressure distributions having the form

$$P = \left(1 - \frac{\eta^2}{2n}\right)^n, \qquad \alpha_i = \frac{2n}{(\xi_i^*)^2} \tag{2.10}$$

All such distributions correspond to parabolic temperature distribution

$$PV = 1 - \frac{\eta^2}{2n} \tag{2.11}$$

According to (2.10) and (2.11) the entropy associated with the particles has the distribution given by

$$PV^\varkappa = \left(1 - \frac{\eta^2}{2n}\right)^{-n(\varkappa-1)+\varkappa}$$

Assuming $n = \varkappa/(\varkappa - 1)$ the distribution given by (2.6) is obtained.

3. In [1] a more general relationship between x_i and ξ_i than that of Equation (1.4) is considered. Namely

$$x_i = \sum_k \varphi_{ik}(t)\, \xi_k, \qquad u_i = \sum_k \xi_k \frac{d\varphi_{ik}}{dt} \tag{3.1}$$

The elements φ_{ik} can be formed into a matrix $M(t)$. Resulting from the definition ξ_k we have $\varphi_{ik}(0) = \delta_{ik}$. Therefore $M(0)$ is a unit matrix. From (1.3) we have

$$v^\circ(t) = |M|\, v^\circ(0) \tag{3.2}$$

Here $|M|$ designates the absolute value of the determinant, with $|M| = 1$ at $t = 0$. Moreover, substituting (3.1) and (1.3) into the second of Equations (1.1) we obtain

$$\sum_k \frac{d^2\varphi_{ik}}{dt^2} \xi_k = -v^\circ(t)\, p^\circ(t) \sum_k \frac{\partial P}{\partial \xi_k} \frac{1}{\varphi_{ik}} \tag{3.3}$$

Equation (3.3) can be satisfied if the variation of the matrix elements φ_{ik} is described by Equation

$$M^\vee \frac{d^2 M}{dt^2} = v^\circ(t)\, p^\circ(t)\, L \tag{3.4}$$

where $M^\vee$ is the transpose of matrix M, and L is a constant square matrix of third order. When only adiabatic expansion is investigated, as in [1], then the following relation is introduced:

$$p^\circ(t) = p^\circ(0)|M|^{-\varkappa} \qquad (3.5)$$

It is evident that Ovsiannikov's solution can be generalized to the non-adiabatic cases as long as energy equations are such as to allow for separation of variables (1.3). In either case, instead of Equation (2.4) we have

$$V\frac{\partial P}{\partial \xi_k} = \sum_l L_{kl}\xi_l$$

The matrix L is skew-symmetric and can always be diagonalized [1]. The solution of the system of equations (3.4) is determined by the initial values of $N(0)$ and $N'(0)$, where $N' = dN/dt$. If the expansion starts from rest then $N'(0)$ is a null matrix. It follows from (3.4) that for this case the matrix N remains diagonal for all time. This is also the case when the initial values of velocities along the axes of coordinates are proportional to the distance from the center of symmetry only along a given coordinate, i.e. when $N'(0)$ is diagonal.

The diagonalization of matrix N is possible if $\varphi_{ik} = \lambda_{ik}\varphi_{ii}$ and, accordingly, instead of (3.1) we shall have

$$x_i = \varphi_{ii}\sum_k \lambda_{ik}\xi_k, \qquad u_i = \frac{d\varphi_{ii}}{dt}\sum_k \lambda_{ik}\xi_k$$

Performing an elementary rotation of coordinates and designating φ_{ii} by φ_i we shall obtain the formulas from Section 1.

In this way it is shown that in the case the motion starts from rest or with initial velocities along the three principal axes of the ellipsoid, then instead of the nine equations (3.4) the system of equations (1.5) should be used, or in the adiabatic case the system (2.3). In the following we shall consider only this case, as it is of greatest practical interest.

4. We shall consider the time variation of the dimensions of the ellipsoid with adiabatic changes of state. We shall introduce a reference time $t_0 = r_0/u_0$, where r_0 is the characteristic dimension and u_0 is the characteristic velocity.

From the system of equations (2.3) we obtain

$$\frac{d^2\varphi_i}{d\tau^2} = \frac{(\varphi_1\varphi_2\varphi_3)^{-(\varkappa-1)}}{\varphi_i}\beta_i^2 \qquad (4.1)$$

$$\left(\beta_i^2 = \alpha_i\frac{r_0^2 p^\circ(0)\, v^\circ(0)}{u_0^2}, \qquad \tau = \frac{t}{t_0}\right)$$

For the distribution of the parameters inside the ellipsoid (2.6) or (2,10), having taken r_0 to be the smallest of the ellipsoid dimensions ξ_1^* (to be specific) and $u_0^2 = 2np^\circ(0)v^\circ(0)$, we obtain $\beta_1^2(\xi_1^*)^2 = (\xi_1^*)^p$

It follows from (4.1) that the greatest variation of φ_i is in the direction of the minor axis of the ellipsoid. If the expansion is one-dimensional or almost one-dimensional then instead of the system (4.1), it is necessary to integrate only one equation [3]

$$\frac{d^2\varphi_1}{d\tau^2} = \varphi_1^{-\nu(\varkappa-1)-1} \qquad (4.2)$$

In the case of spherical symmetry $\nu = 3$, for cylindrical symmetry around the major axis $\nu = 2$, and for a strongly oblated ellipsoid $\nu = 1$, Equation (4.2) is integrable [3].

$$\psi_1^2 = \psi_1^2(0) + \left(1 - \varphi_1^{-\nu(\varkappa-1)}\right)\frac{2}{\nu(\varkappa-1)} \qquad \left(\psi_1 = \frac{d\varphi_1}{d\tau}\right) \tag{4.3}$$

For large values of τ and, consequently large φ_1, the nondimensional velocity ψ approaches a constant value (i.e. inertial expansion follows). The maximum velocity of particles along a given axis, i.e. the velocity of the ellipsoid boundary, is given by

$$u_i^* = \xi_i^* \frac{d\varphi_i}{dt} = \frac{\xi_i^*}{r_0} u_0 \frac{d\varphi_i}{d\tau} = u_0 \frac{\xi_i^*}{\xi_1^*} \psi_i \tag{4.4}$$

The following table gives values of the ratios of the ellipsoid semiaxes at the time when the expabsion becomes inertial, for a number of different initial relative dimensions ξ_i^*/ξ_1^* and different values of the adiabatic exponent $\varkappa$. The results are based on the assumptions of symmetry around one axis of ellipsoid which is initially at rest (i.e. the initial velocities $\psi_i(0) = 0$). The following notation is also introduced: $\xi_1^* s_0 = \xi_3^*$, and $x_3^* = s_1 x_1^*$ at $\tau \to \infty$

	s_0	2	3	5	7	10	$\varkappa$
$x_1 = x_2$	s_1	1.41	1.76	2.39	2.96	3.50	$^7/_5$
		1.55	2.05	2.99	3.87	5.18	$^5/_3$
		1.81	2.62	4.26	5.91	8.39	3
$x_2 = x_3$	s_1	1.41	1.64	2.06	2.41	2.63	$^7/_5$
		1.50	1.88	2.47	2.94	3.56	$^5/_3$
		1.75	2.44	3.73	5.00	6.88	3

It follows from the above table that the ratios of the axes of the ellipsoid change during the process of expansion, in such a manner, that the originally smallest ratio becomes the largest. This results from the fact that the velocity along the major axis is smaller than that along the minor, because, as follows from (4.1), acceleration along the major axis takes place at later stages of the expansion when the gas has already been cooled by expanding in direction of the minor axis of the ellipsoid. The motion in the "secondary" direction, for sufficiently large magnitude of the axes ratio, can't substantially influence the expansion in the "primary" direction, since the latter becomes practically inertial.

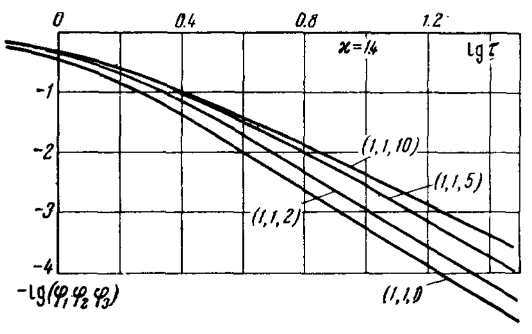

Fig. 1

The table also shows that the effect of variation of the axes ratio becomes more pronounced as the adiabatic exponent $\varkappa$ increases. It is also more pronounced for "cylindrical" ($x_1 = x_2$), than for "disk like" ($x_2 = x_3$) "explosion". This is due to the rate of cooling of the gas. Naturally, the effect also increases with the increase of the initial ratio of the semiaxes, since the interval of time during which the motion is practically one-dimensional, increases.

Fig.1 shows thw time variation of density at the center of the expanding region for a number of different ratios of the ellipsoid semiaxes s_0 and for $\varkappa = 1.4$. With increasing s_0 the density decreases less rapidly, nevertheless for a sufficiently large value of time $\rho \sim t^{-3}$.

We shall discuss the mechanical effect of the directional character of the expansion in terms of the velocity head momentum. Let us compare the magnitude of the momentum at corresponding points for an ellipsoid and a sphere, diameter of which is equal to the minor axis of the ellipsoid. In the case of the ellipsoid the point lies in the central plane normal to the major axis at a distance R from the center. At the instant t_0, when the boundary of the expanding gas reaches this point, the density of the gas at the center becomes

$$\rho\,(0,\,t) = \rho\,(0,\,0)\,\frac{x_1^*\,(0)\,x_2^*\,(0)\,x_3^*\,(0)}{x_1^*\,(t_0)\,x_2^*\,(t_0)\,x_3^*\,(t_0)} \tag{4.5}$$

Let $x_1 = x_2$ and $x_1^* \to s_1 x_3^*$, where s_1 can be determined from the table. Using the notation $x_1^*(0) = r = \xi_1^*$ and bearing in mind that $x_3(t_0) = R$, we obtain from (4.5)

$$R^3\rho\,(0,\,t) = \rho\,(0,\,0)\,s_1 s_0 r^3 \tag{4.6}$$

The velocity along the axes can be found by s_1, taking into account the integral of the system (2.3) cr (4.1)

$$\sum_i \frac{\psi_i^2}{\beta_i} = \frac{2}{(\varkappa-1)}\,[1-(\varphi_1\varphi_2\varphi_3)^{-(\varkappa-1)}] + \sum_i \frac{\psi_i^2\,(0)}{\beta_i} \tag{4.7}$$

For $\varphi_1 = \varphi_2 = \varphi_3$, which is the case for $\beta_1 = \beta_2 = \beta_3$, from (4.7) results (4.3). The existance of such an integral is readily verified by substitution in the initial system.

If $\psi_i(0) = 0$ for all t, using (4.4) and for $\varphi_1\varphi_2\varphi_3 \to \infty$ we obtain

$$\sum_i (u_i^*)^2 = u_0^2\,\frac{2}{(\varkappa-1)} \tag{4.8}$$

For the symmetric case $u_1^* = u_2^*$ during inertial expansion $u_1^* = s_1 u_3^*$; therefore (4.8) leads to

$$(u_1^*)^2 = \frac{2}{(\varkappa-1)}\,\frac{u_0^2}{(2+s_1^{-2})} \tag{4.9}$$

Accordingly, the velocity head momentum is proportional to

$$\frac{\rho\,(0,\,0)\,u_0}{R}\,\frac{s_0 s_1}{\sqrt{2+s_1^{-2}}}$$

This shows that the velocity head momentum for the ellipsoid increases with increasing ratio of the semiaxes and is $s_0 s_1$ times larger than for an "inscribed" sphere.

5. We shall consider the motion of gas with the inclusion of heating. Let the heating intensity be an exponential function of the temperature and the density and an arbitrary function of the time and the Lagrangian space coordinates

$$\frac{\partial e}{\partial t} + p\frac{\partial v}{\partial t} = Q\left(\frac{pv}{p_* v_*}\right)^{-\alpha}\left(\frac{v}{v_*}\right)^{-\beta}\eta\left(\frac{t}{t_*}\right)\lambda(\xi_k) \tag{5.1}$$

As in the one-dimensional case [4 and 5] the variables are separable for some special values of the initial conditions. From (1.3) with (1.6), we obtain an equation for pressure, density in the center and for distribution particle properties

$$\frac{1}{(\varkappa-1)}\frac{d(p^\circ v^\circ)}{dt} + p^\circ\frac{dv^\circ}{dt} = Q\left(\frac{p^\circ v^\circ}{p_* v_*}\right)^{-\alpha}\left(\frac{v^\circ}{v_*}\right)^{-\beta}\eta\left(\frac{t}{t_*}\right) \tag{5.2}$$

$$PV^\gamma = \lambda^{-1/(\alpha+1)}, \qquad \gamma = 1 + \beta / (1+\alpha) \tag{5.3}$$

Naturally, the function $\lambda(\xi_k)$ is subjected to the same restrictions as $f(\xi_k)$ (Equation (2.8)). Distribution of properties can be determined as in Section 2 or as in [5].

We shall consider the case of $\lambda = 1$, (energy release per unit mass is the same for all particles). We shall limit the analysis to the case where $\alpha = 0$, $\beta = 0$, $\gamma = 1$. From (5.2) and (2.3) we get

$$PV = 1,\ P = \exp\left[-{}^1\!/_2\left(\alpha_1\xi_1^2 + \alpha_2\xi_2^2 + \alpha_3\xi_3^2\right)\right] \tag{5.4}$$

For $\gamma = 1$, the region ccupied by the gas is unbounded, and the magnitudes α_i can be determined by e.g. from given dimensions of ellipsoidal surface with a fixed value of the dimensionless pressure $p^\circ < 1$ (we shall note that the mass of the expanding matter is bounded and constant). Let us denote these dimensions by ξ_i^*. Then $(\xi_i^*)^2\alpha_i = 2$. Let $\eta = 1$, i.e. the energy source is of uniform intensity. Then, the solution of the energy equation has an exponential form

$$p^\circ v^\circ = Bt^n p_* v_*, \quad v^\circ = A^\nu t^m v_*, \quad B^{1+\alpha} = \frac{Q(\varkappa-1)A^{-\nu\beta}}{[n + m\nu(\varkappa-1)]\,p_* v_*}$$
$$n(1+\alpha) = 1 - m\nu \tag{5.5}$$

The system of equations determining φ assumes the form

$$\varphi_i\frac{d^2\varphi_i}{dt^2} = \alpha_i B t^m p_* v_* \tag{5.6}$$

These equations have also solutions of exponential form for $\varphi_i \gg 1$

$$x_i = \xi_i\varphi_i = \left(\frac{\xi_i}{\xi_i^*}\right)t^{1+{}^1\!/_2 n}\left(\frac{2Bp_* v_*}{n(1+{}^1\!/_2 n)}\right)^{1/2} \tag{5.7}$$

Particles, whose initial coordinates were $\xi_i = \xi_i^*$, corresponding to $p = p^\circ$, are now fcund at points equidistant from the center. This means that the expansion assumes the spherical symmetry. Similar results are obtained for cases where $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 1$, and when the velocity $\psi_i = d\varphi_i/d\tau$ increases with time without bounds i.e. for $n > 0$. Since it follows from (5.7) that $m = \nu(1 + \frac{1}{2}n)$, then $n(1 + \alpha + \frac{1}{2}\nu\beta) = 1 - \nu\beta$ and for $\alpha > 0$, $\beta > 0$ the index $n > 0$ only for $\beta < 1/\nu$.

The detailed discussion of the character of temperature variation for the one-dimensional case is given in [5].

The exact solutions of the equations of gas dynamics were obtained for particular values of the initial conditions. The results can be also used to describe approximately the variation of parameters at the initial conditions slightly different from those for which the expansion is selfsimilar at the initial stage.

It must be noted, however, that the particular solution considered shall not be asymptotic for the adiabatic case. The case of expansion with heating and unlimited temperature rise will give the asymptote (for physical reasons [4 and 5]) for various initial distributions of the density (as was shown for the one-dimensional case [5]) and for arbitrary shape of the initially cold cloud. However, this should be verified by numerical solutions.

For the adiabatic cases, or cases with heating but with falling tempetatures, the differences in the patterns of motion for various conditions can be not only quantitative, but also qualitative. E.g. for $\varkappa > 2$, for sufficiently long cylinder with constant parameters along the axis, there exists a region where the motion is strictly cylindrical, as following the expansion in direction normal to the axis and rapid decrease in sonic velocity, the influence of the ends is not felt in the central region [5] where the density follows the law $\rho \sim t^{-2}$. However, in the considered example of ellipsoid expansion with pressure gradients in the axial direction at the initial stage of the process, after all, follows the cubic law.

The author is indebted to O.S.Ryzhov and G.M.Shefter for valuable discussions and to A.N.Zimina for computations.

BIBLIOGRAPHY

1. Ovsiannikov, L.V., Novoe reshenie uravnenii girodinamiki (A new solution of the equations of hydrodynamics). Dokl.Akad.Nauk SSSR, Vol.111, №1, (pp.47-49), 1956.
2. Sedov, L.I., Ob integrirovanii uravnenii odnomernogo dvizheniia gaza (On the integration of the equations of uniform motion of gas). Dokl.Akad.Nauk SSSR, Vol.90, № 5, (p.735), 1953.
3. Sedov, L.I., Metody podobiia i razmernosti v mekhanike (Methods of Similitude and Dimensional Analysis in Mechanics). Gostekhizdat, Izd.3, (pp.242-248), 1954.
4. Nemchinov, I.V., Razlet ploskogo sloia gaza pri postepennom vydelenii energii (Expansion of a plane gas layer with gradual energy release). PMTF, № 1, (pp.17-26), 1961.
5. Nemchinov, I.V., Razlet podogrevaemoi massy gaza v reguliarnom rezhime (Expansion of a heated gas in a regular behavior). PMTF, № 5, 1964.

Translated by M.J.